THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2017)
Tutorial 12

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1. (a) Let $\{x_n\}, \{y_n\}$ be sequences of complex numbers and let $s_n := \sum_{k=1}^n y_k$, $s_0 := 0$. Prove the summation by parts formula:

$$\sum_{k=n+1}^{m} x_k y_k = (x_m s_m - x_{n+1} s_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) s_k,$$

for $m > n$.

(b) Use summation by parts formula to prove the Kronecker's Lemma: Let

$$\sum_{n=1}^{\infty} x_n = s \in \mathbb{R}$$

Let $0 < b_1 \leq b_2 \leq \cdots \leq b_n \to \infty$. Then

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^{n} b_k x_k = 0.$$

Note in particular, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k = 0.$$

(c) Use summation by parts formula to study the convergence of the series:

$$\sum_{n=1}^{\infty} \frac{\cos \pi n}{n}$$

Hint: Write $\cos(\pi n) = \mathrm{Re}(e^{i\pi n})$.

(d) Use summation by parts formula to compute:

$$\sum_{j=1}^{\infty} j \left(\frac{1}{2}\right)^j.$$

2. These are for reference only.

Theorem 1. *Suppose* $\{a_n\}$ *are complex numbers so that the power series*

$$G(x) := \sum_{n=0}^{\infty} a_n x^n$$

has radius of convergence 1.

(a) (Abel's Theorem) Suppose

$$\sum_{n=0}^{\infty} a_n = s \in \mathbb{C}.$$

Then $\lim_{x \to 1^+} G(x) = s$.

(b) (Littlewood's theorem) Conversely, suppose $\lim_{x \to 1^+} G(x) = s \in \mathbb{C}$, *and that there is* $C > 0$ *such that for any* $n \in N$,

$$|na_n| \leq C$$

Then

$$\sum_{n=0}^{\infty} a_n = s.$$